We have seen that the graph of a function is a powerful tool for getting information about the function. In this section we’ll use graphs of functions to help us solve inequalities. First notice that from the graph of a function \( f \) we can easily determine the solution of the inequality

\[
f(x) > 0
\]

The solutions of this inequality consist of all those values of \( x \) for which the graph of \( f \) is above the \( x \)-axis. So we can find the solutions by inspecting the graph of \( f \) to determine the intervals on the \( x \)-axis on which the graph of \( f \) is above the \( x \)-axis. The endpoints of such intervals are the points where

\[
f(x) = 0
\]

From the graph of \( f \) in Figure 1 we see that the solutions of the inequality \( f(x) > 0 \) are the intervals \((-\infty, -3)\) and \((2, 7)\).

In Section 2.3 we learned how to solve inequalities graphically, by using a graph obtained from a graphing calculator. Here we use both algebra and the graph to solve inequalities.
Linear Inequalities and Graphs

Recall that an inequality is called linear if it can be expressed in the form

\[ ax + b \geq 0 \]

where the symbol \( \geq \) can be replaced by any of the inequality symbols \( (\geq, >, \leq, <) \). The following example illustrates the method outlined above for solving linear inequalities.

**EXAMPLE 1 | Solving Linear Inequalities by Using a Graph**

Solve the inequality.

(a) \(-\frac{1}{2}x + 1 \leq 0\)

(b) \(-\frac{1}{2}x + 1 < 0\)

(c) \(-\frac{1}{2}x + 1 > 0\)

**SOLUTION**

(a) We follow the steps in the preceding example..

**Solve the corresponding equation.** We first solve the equation

\[-\frac{1}{2}x + 1 = 0\]

obtained by replacing the inequality sign with an equal sign. We see that the solution is \( x = 2 \). So the number 2 divides the real line into the two intervals \((-\infty, 2)\) and \((2, \infty)\) as shown in Figure 2.

**Graph the appropriate function.** We express the left-hand side of the inequality as a function:

\[ f(x) = -\frac{1}{2}x + 1 \]
The graph of the function $f$ is the line shown in Figure 3. Note that the x-intercept of the graph is 2.

![Figure 3](image)

**Figure 3** Graph of $f(x) = -\frac{1}{2}x + 1$

**Inspect the graph and determine the solution of the inequality.**

Since we want to solve the inequality

$$f(x) = -\frac{1}{2}x + 1 \leq 0$$

we inspect the graph of $f$ to determine the values of $x$ for which the graph is below or on the x-axis. From the graph we see that this occurs for $x$ in the interval $[2, \infty)$. So the solution of the inequality is the interval $[2, \infty)$.

(b) Since we want to solve the inequality

$$f(x) = -\frac{1}{2}x + 1 < 0$$

we inspect the graph of $f$ in Figure 3 to determine the values of $x$ for which the graph of $f$ is strictly below (but not on) the x-axis. From the graph we see that this occurs for $x$ in the interval $(2, \infty)$. So the solution of the inequality is $(2, \infty)$.

(c) Since we want to solve the inequality

$$f(x) = -\frac{1}{2}x + 1 > 0$$

we inspect the graph of $f$ in Figure 3 to determine the values of $x$ for which the graph is strictly above (but not on) the x-axis. From the graph we see that this occurs for $x$ in the interval $(-\infty, 2)$. So the solution of the inequality is $(-\infty, 2)$.

Now try Exercise 1
Quadratic Inequalities and Graphs

Recall that an inequality is called quadratic if it can be expressed in the form

\[ ax^2 + bx + c \geq 0 \]

where the symbol “\( \geq \)” can be replaced by any of the inequality symbols (\( \geq, >, \leq, < \)). The following example illustrate the method outlined above for solving quadratic inequalities.

**EXAMPLE 2** | Solving Quadratic Inequalities by Using a Graph

Solve the inequality:

(a) \( x^2 - x - 6 > 0 \)

(b) \( x^2 - x - 6 \leq 0 \)

**SOLUTION**

(a) We follow the steps in the preceding examples.

**Solve the corresponding equation.** We first solve the equation

\[ x^2 - x - 6 = 0 \]

obtained by replacing the inequality sign with an equal sign. To solve this equation we factor the left-hand side:

\[ x^2 - x - 6 = 0 \text{ Equation} \]

\[ (x + 2)(x - 3) = 0 \text{ Factor} \]

\[ x = -2 \text{ or } x = 3 \]

We see that the solutions are \( x = -2 \) and \( x = 3 \). The numbers \(-2\) and \(3\) divide the real line into the three intervals \((-\infty, -2)\), \((-2, 3)\), and \((3, \infty)\) as shown in Figure 4.

**Figure 4** The intervals \((-\infty, -2)\), \((-2, 3)\), and \((3, \infty)\)

**Graph the appropriate function.** We express the left-hand side of the inequality as a function:

\[ f(x) = x^2 - x - 6 \]
The graph of the function \( f \) is the curve shown in Figure 5. Note that the \( x \)-intercepts of the graph are \(-2\) and \(3\).

![Graph of \( f(x) = x^2 - x - 6 \)](image)

**Figure 5** Graph of \( f(x) = x^2 - x - 6 \)

**Inspect the graph and determine the solution of the inequality.**

Since we want to solve the inequality

\[
f(x) = x^2 - x - 6 > 0
\]

we inspect the graph of \( f \) to determine the values of \( x \) for which the graph is above (but not on) the \( x \)-axis. From the graph we see that this occurs for \( x \) in the intervals \((-\infty, -2)\) and \((3, \infty)\). So the solution of the inequality is \((-\infty, -2) \cup (3, \infty)\).

**(b)** Since we want to solve the inequality

\[
f(x) = x^2 - x - 6 \leq 0
\]

we inspect the graph of \( f \) in Figure 5 to determine the values of \( x \) for which the graph of \( f \) is strictly below or on the \( x \)-axis. From the graph we see that this occurs for \( x \) in the interval \([-2, 3]\). So the solution of the inequality is \([-2, 3]\).

Now try Exercise 7
Polynomial Inequalities and Graphs

An inequality is called polynomial inequality if it can be expressed in the form

\[ p(x) \geq 0 \]

where \( p \) is a polynomial expression; the symbol \( \geq \) can be replaced by any of the inequality symbols (\( \geq, >, \leq, < \)). The following example illustrates the method outlined above for solving polynomial inequalities.

**EXAMPLE 3**  |  Solving Polynomial Inequalities by Using a Graph

Solve the inequality:

(a) \( x^3 - 8x^2 < 0 \)
(b) \( x^3 - 8x^2 > 0 \)
(c) \( x^3 - 8x^2 \geq 0 \)

**SOLUTION**

(a) We follow the steps in the preceding examples.

**Solve the corresponding equation.** We first solve the equation \( x^3 - 8x^2 = 0 \). We solve this equation algebraically:

\[
\begin{align*}
x^3 - 8x^2 &= 0 \\
x^2(x - 8) &= 0 \\
x &= 0 \quad \text{or} \quad x = 8
\end{align*}
\]

We see that the zeros are \( x = 0 \) and \( x = 8 \). The numbers 0 and 8 divides the real line into the three intervals \(( -\infty, 0 ), (0, 8), \) and \((8, \infty)\) as shown in Figure 6.

![Figure 6: The intervals \(( -\infty, 0 ), (0, 8), \) and \((8, \infty)\)](image)

**Graph the appropriate function.** We express the left-hand side of the inequality as a function:

\[ f(x) = x^3 - 8x^2 \]
The graph of the function \( f \) is the line shown in Figure 7. Note that the **x-intercepts** of the graph are 0 and 8.

**Figure 7** Graph of \( f(x) = x^3 - 8x^2 \)

**Inspect the graph and determine the solution of the inequality.**

Since we want to solve the inequality

\[
f(x) = x^3 - 8x^2 < 0
\]

we inspect the graph of \( f \) to determine the values of \( x \) for which the graph is below (but not on) the \( x \)-axis. From the graph we see that this occurs for \( x \) in the intervals \((-\infty, 0) \) and \((0,8) \). So the solution of the inequality is

\[
(-\infty, 0) \cup (0,8)
\]

Notice that we excluded 0 from the solution because the strict inequality is not satisfied when \( x \) is 0.

(b) Since we want to solve the inequality

\[
f(x) = x^3 - 8x^2 > 0
\]

we inspect the graph of \( f \) in Figure 7 to determine the values of \( x \) for which the graph is strictly above (but not on) the \( x \)-axis. From the graph we see that this occurs for \( x \) in the interval \((8, \infty) \). So the solution of the inequality is \((8, \infty) \).

(c) Since we want to solve the inequality

\[
f(x) = x^3 - 8x^2 \geq 0
\]

we inspect the graph of \( f \) in Figure 7 to determine the values of \( x \) for which the graph is above or on the \( x \)-axis. From the graph we see that this occurs for \( x \) in the interval \([8, \infty) \) and also when \( x \) is 0. So the solution of the inequality is \( \{0\} \cup [8, \infty) \).

Now try Exercise 13
A rational inequality is an inequality of the form
\[ \frac{p(x)}{q(x)} \geq 0 \]
where \( p \) and \( q \) are polynomials; the symbol “\( \geq \)” can be replaced by any of the inequality symbols (\( \geq, >, \leq, < \)). The following example illustrates the method outlined above for solving rational inequalities.

**EXAMPLE 1**  |  Solving Rational Inequalities by Using a Graph

(a) \[ \frac{x^2 - 1}{x^2 - 4} < 0 \]  
(b) \[ \frac{x^2 - 1}{x^2 - 4} \geq 0 \]

**SOLUTION**

(a) We use the following steps to solve the inequality:

**Solve the corresponding equation and find the zeros of the denominator.** Let’s consider the equation
\[ \frac{x^2 - 1}{x^2 - 4} = 0 \]
The solutions of this equation are the values of \( x \) which make the numerator 0.
\[ x^2 - 1 = 0 \quad \text{Set numerator equal to 0} \]
\[ (x + 1)(x - 1) = 0 \quad \text{Factor} \]
\[ x = -1 \quad \text{or} \quad x = 1 \]

Next we find the zeros of the denominator:
\[ x^2 - 4 = 0 \quad \text{Set denominator equal to 0} \]
\[ (x + 2)(x - 2) = 0 \quad \text{Factor} \]
\[ x = -2 \quad \text{or} \quad x = 2 \]
The numbers \(-1, 1, -2, 2\) divide the real line into the five intervals \((-\infty, -2), (-2, -1), (-1, 1), (1, 2)\) and \((2, \infty)\) as shown in Figure 8.

**Graph the appropriate function.** We express the left-hand side of the inequality as a function:
The graph of the function \( f \) is the line shown in Figure 9. Note that the \textbf{x-intercepts} of the graph are \(-1\) and \(1\) and the graph has \textbf{vertical asymptotes} \( x = -2 \) and \( x = 2 \).

\[
f(x) = \frac{x^2 - 1}{x^2 - 4}
\]

**Figure 9** Graph of \( f(x) = (x^2 - 1)/(x^2 - 4) \)

**Inspect the graph and determine the solution of the inequality.**

Since we want to solve the inequality

\[
f(x) = \frac{x^2 - 1}{x^2 - 4} < 0
\]

we inspect the graph of \( f \) to determine the values of \( x \) for which the graph is \textit{below} (but not on) the \( x \)-axis. From the graph we see that this occurs for \( x \) in the intervals \((-2, -1)\) and \((1, 2)\). So the solution of the inequality is \((-2, -1) \cup (1, 2)\).

(b) Since we want to solve the inequality

\[
f(x) = \frac{x^2 - 1}{x^2 - 4} \geq 0
\]

we inspect the graph of \( f \) in Figure 9 to determine the values of \( x \) for which the graph is \textit{above and on} the \( x \)-axis. From the graph we see that this occurs for \( x \) in the intervals \((-\infty, -2)\), \([-1, 1]\), and \((2, \infty)\). So the solution of the inequality is \((-\infty, -2) \cup [-1, 1] \cup (2, \infty)\).

Note that \( f \) is not defined at \(-2\) and \(2\), so these points are not in the domain of \( f \).

Now try Exercise 19
**EXERCISES**

1–4 ■ Solve the linear inequality. Express the answer in interval notation.

1. $2x - 6 \geq 0$
2. $2x - 6 > 0$
3. $\frac{1}{3}x + 2 < 0$
4. $\frac{1}{3}x + 2 \leq 0$
5. $4x > 50$
6. $-2x < 7$

7–12 ■ Solve the quadratic inequality. Express the answer in interval notation.

7. $x^2 - 2x - 8 < 0$
8. $x^2 - 2x - 8 > 0$
9. $x^2 - 5x + 6 \geq 0$
10. $x^2 - 5x + 6 \leq 0$
11. $2x^2 + 3x > 2$
12. $2x^2 < 5x + 3$

13–18 ■ Solve the polynomial inequality. Express the answer in interval notation.

13. $3x^2 - x^3 \leq 0$
14. $3x^2 - x^3 < 0$
15. $x^3 - 2x^2 + x > 0$
16. $x^3 + 6x^2 + 9x \geq 0$
17. $x^4 < 4x^2$
18. $x^4 \leq 8x^2 - 16$

19–24 ■ Solve the rational inequality. Express the answer in interval notation.

19. $\frac{x - 2}{x + 3} > 0$
20. $\frac{2x + 6}{x - 4} < 0$
21. $\frac{x^2 - 4}{x^2 - 2x + 1} \leq 0$
22. $\frac{6x}{x^2 - 9} \geq 0$
23. $\frac{2x - 8}{x^2 - 1} < 0$
24. $\frac{x^2 - 4}{x^2 - 1} \leq 0$