Simple Harmonic Motion of Springs

Purpose

In this laboratory we will measure the elastic force exerted by a spring and its period of motion when stretched by an attached mass. We will also determine the effective mass of the vibrating mass-spring system and develop an empirical model for its period of oscillation.

Principles

Springs are examples of elastic systems that resist any deformation of their shape. When a spring is stretched or compressed by an external force, it will push or pull against the force in an attempt to return to its equilibrium length. This interplay between an external force – a weight hung from the spring, for instance – and the restoring spring force leads to an oscillatory motion known as simple harmonic motion.

Simple harmonic motion is motion that repeats itself in a sinusoidal fashion. That is, the motion is cyclic, and the graph of the cycles over time are sine curves. If we know how the motion takes place during one cycle, we know the motion for all time (for an ideal, frictionless system), since succeeding cycles are the same as the first.

Many systems in nature have similar elastic properties. In fact, the molecules in any solid behave to a good first approximation as if they were bound together by springs, so the principles underlying the behavior of springs have quite general application.

Linear Restoring Forces

The physicist Robert Hooke first investigated the spring force systematically in the 17th Century. Hooke found that the restoring force exerted by a spring upon being stretched or compressed was proportional to the elongation or compression of the spring. This can be expressed as

\[ F_s = -kx \]

where \( F_s \) is the spring force, \( x \) is the elongation or compression and \( k \) is the constant of proportionality which measures the stiffness of the spring. The above expression is known as Hooke’s Law.

This kind of force is known as a linear restoring force. A linear restoring force will lead to simple harmonic motion. For instance, if we attach a mass to a horizontal spring on a frictionless surface and stretch it out, the spring will pull back on the mass with a force
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that is proportional to the amount of stretch.  (See Diagram 1, where the unstretched position is indicated by $x = 0$).

If we then release the mass, it will be accelerated towards the equilibrium position, $x = 0$, where the restoring force of the spring and the displacement of the mass are both zero. Since the mass develops a velocity during this process, it will overshoot the equilibrium position and compress the spring. The spring will resist this compression, again with a force proportional to the amount of compression, so that the mass will be slowed to a stop and then accelerated toward equilibrium again. It will again overshoot, and the process will repeat itself.

![Diagram 1](image)

We can graph the force on the mass as a function of displacement from equilibrium as follows:

**Force v. Displacement**

![Force v. Displacement Graph](image)
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From the graph, we see that a linear restituting force has the following general form:

\begin{equation}
F = -kx
\end{equation}

where \(x\) is the displacement from equilibrium and \(k\) is the magnitude of the slope of the graph. For the mass-spring system, \(k\) is the stiffness of the spring, but other parameters play the role of \(k\) for other systems.

**Angular frequency**

A key parameter of the motion is its period: the time for the system to go through one cycle. Newton’s Laws of Motion make a definite prediction for the period of any such system. If the force is as above, then by Newton’s 2nd Law, \(F = ma\), we have

\[-kx = ma\]

which means that

\[a = -\frac{k}{m}x\]

giving us the acceleration of the system at any displacement \(x\). It is convenient to define a positive constant \(\omega^2\) as:

\begin{equation}
\omega^2 \equiv \frac{k}{m}
\end{equation}

so that the acceleration can be written

\[a = -\omega^2 x .\]

\(\omega\) is called the angular frequency of the system. It measures how quickly the system goes through its cycle. The units of \(\omega\) are radians per second, or sec\(^{-1}\), also known as hertz. The period of motion of the system can be determined from the angular frequency, as explained below.

**Equation of Motion**

Let us find the equation of motion for the system. That is, we need to find the displacement \(x\) as a function of time: \(x(t)\).
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Since the acceleration of the system is the second derivative of \( x(t) \), and Newton’s Law has specified the acceleration as above, we can write:

\[
\frac{d^2x}{dt^2} = -\omega^2 x
\]

To solve the above equation, we need to find a function \( x(t) \) that has the property that the equation specifies: that the second derivative of the function is equal to the function itself multiplied by \(-\omega^2\).

It is shown in Calculus that both the sine function and the cosine function have this property, and that a solution to the above differential equation is either

\[
(4a) \quad x(t) = x_0 \sin(\omega t) \quad \text{or}
\]
\[
(4b) \quad x(t) = x_0 \cos(\omega t)
\]

where \( x_0 \) is the displacement at time zero.

You can confirm that either of the above functions are solutions by taking their second derivatives. In our case, the solution using the cosine function is appropriate, since the mass-spring system will start at time zero with a non-zero displacement. (That is, \( x(t = 0) = x_0 \), where \( x_0 \) is the initial displacement of the system. Since the system will periodically return to \( x_0 \), and since it is the largest displacement of the system in absolute value, \( x_0 \) is also called the amplitude of the motion.)

Period of the Motion

We can use the solution to find the period \( T \) of oscillation. At time zero, the system is located at \( x_0 \); at time \( T \) the system has returned to \( x_0 \). Since the equation of motion (4b) tells us that \( x_0 \cos(\omega T) = x_0 \), we see that we must have:

\[
\cos(\omega T) = 1
\]

which means that:

\[
\omega T = 2\pi
\]

or

\[
T = \frac{2\pi}{\omega}
\]
which is the general formula relating the period of an oscillating system to its angular frequency. Applied to the mass-spring system we have

\[ T = 2\pi \sqrt{\frac{m}{k}} \]

**Real Springs**

The above applies to a massless spring. Our real spring has some mass \( m_s \), and this should be included in the mass of the oscillating system. In Experiment 2 we will determine the *effective mass* of the system, so that we can write the period formula as

\[ T = 2\pi \sqrt{\frac{m_{\text{eff}}}{k}} \]

(7)

Since the loops of the spring do not all oscillate to the same degree (one end of the spring is displaced nearly as much as the attached mass, but the other end of the spring is nearly motionless), we will express the effective mass of the system as

\[ m_{\text{eff}} = m + q m_s \]

where \( m \) is the attached mass and \( q \) is a fractional parameter \( (0 < q < 1) \) that determines how much of the mass of the spring contributes to the effective mass.

To analyze our real spring, then, we need to determine two parameters: the spring stiffness \( k \) and the effective mass parameter \( q \). In Experiment 1, below we will determine \( k \) by hanging masses from a vertical spring and measuring the resulting elongation. If the elongation is indeed proportional to the weight of the hanging mass, we will have confirmed Hooke’s Law and the stiffness \( k \) will be the ratio \( F_s/x \).

In Experiment 2, we will measure the period of oscillation for selected masses. By graphing our results, we will determine the parameter \( q \). With \( k \) and \( q \) determined, we can make and test predictions of the period of the system.
Experiment 1: The Spring Constant

Here we will determine the force constant \( k \) of the spring. If we hang the spring vertically, the spring will stretch because of its own weight. The position where the lower end of the spring comes to rest without oscillating is the equilibrium position for the vertical spring. If we hang some additional mass on the spring, the spring will stretch out until the restoring force of the spring \( F_s \) balances the weight \( W \) of the mass. At the new equilibrium position we have:

\[
F_s = -kx = -W
\]

where \( x \) is the elongation of the spring because of the weight of the hanging mass. In the above, \( F_s \) is directed upward while \( x \) and \( W \) are directed downward. Thus

(8) \[ W = kx \]

If we hang several weights and measure the elongation for each, we can graph the above equation. The slope of the graph will be the spring constant \( k \).

To hang masses from the spring, we need to use a mass hanger. Since we are only interested in the elongation of the spring that results when extra weight is hung from it, it will be convenient to treat the hanger as part of the spring. The elongation \( x \) can be measured from the bottom of the hanger; and the extra weight is the weight on the hanger, not including the weight of the hanger.

Equipment

- Table clamp
- Support rod
- Pendulum clamp
- Spring
- Mass hanger
- Masses: 4 100-g and 1 50-g
- Pendulum bob
- String
- Meter stick

Procedures

1. Using a support rod and pendulum clamp, hang the spring vertically about 1.5 meters from the floor. Hang a 50-g mass hanger from the spring.
2. Tape a 2-meter stick next to the vertically suspended spring and hanger so you can measure the position of the bottom of the hanger. Record the position of the bottom of the hanger as \( x_0 \).
3. Hang 100, 200, 300, 400, and 500 grams in succession from the hanger. Let the spring stretch out slowly so that it does not oscillate. Record the masses. Record the new position of the bottom of the hanger as $x_i$.

**Analysis**

1. Calculate the weight of the hanging masses and record these values. Include the case of zero mass (no extra mass on hanger) in your analysis.
2. Calculate the elongation of the spring for each weight: $x = x_i - x_0$. For the zero mass case, $x_i = x_0$ so the elongation $x = 0$. Tabulate your data in the table below.
3. Graph the elongation ($x$) as a function of the weight suspended from the spring. Draw the best-fit line through your data points.
4. Determine the spring constant $k$ from the slope of the graph. Write down the equation of the graph.
Experiment 2: Period of Oscillation

Now we will develop a model for the mass-spring system. We will hang several masses from the vertical spring and measure the period of the system when it is set in motion. We will graph the squared period as a function of hanging mass. We can determine the effective mass of the spring from a graph of this data.

As described in the Principles section, the mass-spring system should vibrate with a period given by

$$T = 2\pi \sqrt{\frac{m_{\text{effective}}}{k}}, \text{ with } m_{\text{effective}} = m + qm_s.$$

If we square the period formula and use the expression for the effective mass we get:

$$T^2 = \frac{4\pi^2}{k} (m + qm_s)$$

If we treat $T^2$ as the dependent variable and $m$ as the independent variable, we can graph the above as a straight line, with $4\pi^2 / k$ as the slope. The term $4\pi^2 km_s / k$ is the y-intercept of the graph. Since $k$ and $m_s$ are known, we can determine the value of $q$ for our spring by reading the y-intercept off our graph and solving for $q$.

Once we have done this, we will have determined all the parameters necessary to describe the motion of the mass-spring system. The formula above will constitute an empirical model for the system.

What would happen if we just set the spring oscillating with no hanging mass? We probably would not find that the spring oscillates with the period described by our empirical model. This is because, for very small masses, Hooke’s Law breaks down and the restoring force on the spring is not proportional to the displacement. Likewise, a very large hanging mass would deform the spring and our model would again break down. Thus we expect our model to be valid only in some middle range of masses: $0 < m < M$.

Procedures

1. Weigh the spring and record the value.
2. Suspend the spring vertically and hang a 50-gram mass hanger from its end.
3. Place 50, 150, 250, 350, 450 grams on the hanger in succession. For each trial, record the total mass hanging from the spring, including the mass hanger. This is $m$ in equation (9). Displace the bottom of the hanger 5 centimeters from equilibrium and then release it so that the system oscillates.
4. Time 20 cycles for each mass. Start your timer as you release the system and count 20 cycles, stopping the timer when the mass returns to the starting position for the 20th time.
5. In your data table, record the time for 20 cycles. These masses and times will be our calibration points.
6. Now time the system with 0, 100, 200 and 400 grams on the hanger, and with 20 grams total hanging mass, again using an initial displacement of 5 centimeters. For the 20 grams, take the hanger off and tie or tape the mass to the spring. We will compare the periods for these masses with the predictions of our model. The purpose of the 20-grams mass is to see if the model holds up for very small masses.
7. To test the idea that the period does not depend on the amplitude of oscillation, time the system with 200 grams on the hanger and initial displacements of 10 cm and 15 cm.

Analysis

1. Calculate the period for each mass by dividing the total time by the number of cycles: 
   \( T = \frac{T_{20}}{20} \).
2. For the masses in the calibration set, square the measured period and record the values in the data table.
3. Graph the squared period as a function of mass. Draw a best-fit line through the data points and extrapolate this line back to \( m = 0 \).
4. From the slope of the graph, calculate the spring constant, \( k \). Compare the value obtained here with that obtained previously.
5. Read the y-intercept from the graph and record this value as \( T^2(0) \).
6. Using equation (9) and your value for \( k \) and \( T^2(0) \), determine the value of \( \theta \). This is our empirical model.
7. Using the model, calculate the periods for each of the masses in the second set. Compare the calculated periods with the measured periods by taking the percent difference for each.
8. Using the model, calculate the period for a 200-gram hanging mass. Compare the actual periods for the 5-, 10-, and 15-cm displacements with the calculated period by taking a percent difference.
Experiment 3: The Time Dependence of the Spring’s Motion

Here we will use DataStudio to plot the position, velocity and acceleration of the spring as it oscillates. From this we can determine the period, the frequency and the angular frequency of the mass-spring system and compare these with our previous results.

Procedures

1. Connect the motion sensor to the Science Workshop interface, and the interface to the computer with the USB cable. Turn on the computer and call up the file, SHM.ds.
2. Hang about 420 grams from hanger. (Use a different mass value than you used in the previous procedures.) Place the motion sensor on the floor directly under the bottom of the mass hanger.
3. Set the masses oscillating with an amplitude of 5-10 cm. Arrange it so that the bottom of the hanger does not come closer to the sensor than 14 cm. (The sensor will not register objects closer than this.)
4. Click “start” on DataStudio. You should get three plots: the position, velocity and acceleration of the masses. The plotting will stop automatically after 10 seconds.
5. Use the “scale to fit” button on each plot to expand the scale. Use the “align axes” button to ensure the time axes are aligned with each other.
6. If you do not get uniform sine wave plots in each graph, delete the data and try again until you do. (To delete data, go to the “Experiment” menu and select “Delete last data run.”)

Analysis

1. Determine the average period of oscillation from the position plot and compare it with the predicted value from your model. Use as many cycles as possible for this.
2. Find the frequency of the system from your plot: that is, how many cycles does the system go through per second?
3. Find the angular frequency, \( \omega \). The angular frequency is just the frequency multiplied by \( 2\pi \). Compare this value to that predicted by your model: \( \omega = \sqrt{\frac{k}{m_{\text{effective}}}} \)
4. Find the angular frequency by a different method: Use the data in the position and acceleration plots to confirm that \( \omega^2 = -a/x \) (see equation (3) above). Remember that \( x \) is the displacement from equilibrium. Since DataStudio measures position from the floor (where the motion sensor was placed), you will need to determine the equilibrium position from your position plot and measure displacement from that point. Measure \( x \) and \( a \) at five different points in time – not all peaks – and compare their average ratio to \( \omega^2 \) from your model.
Data: Spring Constant

**Note:** Your data should be recorded in your lab notebook. The following is a guide only.

**Experiment 1: Spring Constant**

\[ x_0 = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \]

<table>
<thead>
<tr>
<th>Mass on hanger</th>
<th>Weight</th>
<th>Position of bottom of hanger ((x_1))</th>
<th>Elongation ((x = x_1 - x_0))</th>
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Slope of graph: ________  Spring Constant \(k\): ________

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Data: Period of Oscillation

Experiment 2: Period of the Mass-Spring System

Mass of spring: __________

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<thead>
<tr>
<th>Calibration points</th>
<th>Amplitude ($x_0$): __________</th>
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<tbody>
<tr>
<td>Mass (m)</td>
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<tr>
<td>Time for 20 cycles (T_{20})</td>
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<tr>
<td>Period (T)</td>
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<td>Period squared ($T^2$)</td>
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</table>

Spring constant: __________

Equation of mass-spring system:
## Data: Period of Oscillation

### Test points

<table>
<thead>
<tr>
<th>Suspended Mass</th>
<th>Initial displacement</th>
<th>Time for 20 vibrations</th>
<th>Period</th>
<th>Percent difference</th>
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