Some Important Series

\[ \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \text{ for } |x| < 1 \]
\[ \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \]
\[ \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \]
\[ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \]
\[ \tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)} \text{ for } |x| \leq 1 \]

Binomial Series

\[ (1 + x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n \text{ for } |x| < 1 \text{ where } \binom{k}{0} = 1 \text{ and } \binom{k}{n} = \frac{k(k-1)\cdots(k-n+1)}{n!} \text{ for } n \geq 1 \]

Geometric Series

\[ \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \text{ for } |r| < 1, \text{ diverges for } |r| \geq 1 \]

Harmonic Series/p-Series

\[ \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges.} \]
More generally \[ \sum_{n=1}^{\infty} \frac{1}{n^p} \text{ diverges if } p \leq 1 \text{ and converges if } p > 1 \]

Power Series Centered at a:

\[ \sum_{n=0}^{\infty} c_n (x-a)^n \]

Taylor Series/Maclaurin Series:

If a function \( f \) has a power series representation at \( a \), then it can be written in the form

\[ f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \]

This is called the Taylor series of \( f \) at \( a \).

If \( a = 0 \), then this is a Maclaurin series: \[ f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \]
Convergence Tests

**n-th Term Test:** \[ \sum_{n=1}^{\infty} a_n \text{ diverges if } \lim_{n \to \infty} a_n \neq 0 \text{ or } \text{DNE} \]

**Integral Test:** If \( f(x) \) continuous, positive, decreasing for \( x \in [1, \infty) \) and \( a_n = f(n) \)
\[ \Rightarrow \text{If } \int_1^{\infty} f(x)dx \text{ converges (diverges), then } \sum_{n=1}^{\infty} a_n \text{ converges (diverges)} \]

\( a_n \) and \( b_n \geq 0 \) for all \( n \), then

**Comparison Test:**
1) \( \sum b_n \) converges and \( a_n \leq b_n \) for all \( n \) \( \Rightarrow \sum a_n \) converges
2) \( \sum b_n \) diverges and \( a_n \geq b_n \) for all \( n \) \( \Rightarrow \sum a_n \) diverges

**Limit Comparison Test:**
If \( a_n, b_n > 0 \) and \( \lim_{n \to \infty} \frac{a_n}{b_n} = c \) \( \neq 0, (c \text{ finite}) \)
then \( \sum a_n \) and \( \sum b_n \) both converge or both diverge

**Alternating Series Test:**
\[ b_n > 0 \]
\[ b_{n+1} \leq b_n \]
\[ \lim_{n \to \infty} b_n = 0 \]
\[ \Rightarrow \sum (-1)^{n-1} b_n \text{ converges} \]

**Absolute Convergence Test:** \( \sum |a_n| \) converges \( \Rightarrow \sum a_n \) converges

**Ratio Test:** Let \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L \). Then
\[ L < 1 \Rightarrow \sum a_n \text{ is absolutely convergent} \]
\[ L > 1 \text{ or } L = \infty \Rightarrow \sum a_n \text{ diverges} \]
\[ L = 1 \Rightarrow \text{inconclusive} \]

**Root Test:** Let \( a_n \geq 0 \) and \( \lim_{n \to \infty} \sqrt[n]{a_n} = L \). Then
\[ L < 1 \Rightarrow \sum a_n \text{ converges} \]
\[ L > 1 \text{ or } L = \infty \Rightarrow \sum a_n \text{ diverges} \]
\[ L = 1 \Rightarrow \text{inconclusive} \]