Calculus III: Formulas: Curvature and Acceleration in 3 Dimensions

\( \mathbf{r}(t) = <x(t), y(t), z(t)> \), \( \mathbf{r}'(t) = \mathbf{v}(t) = <x'(t), y'(t), z'(t)> \), \( \mathbf{v}(t) = |\mathbf{v}(t)| \),
\( \mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) \)

1. **Arc Length** \( s \): \( s(t) = \int_a^t \sqrt{(x'(u))^2 + (y'(u))^2 + (z'(u))^2} \, du = \int_a^t |\mathbf{r}'(u)| \, du \)
   a. From \( t = a \) to \( t = b \): \( s = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} \, dt = \int_a^b |\mathbf{v}(t)| \, dt \)
   c. So \( \frac{ds}{dt} = |\mathbf{r}'(t)| = |\mathbf{v}(t)| \)
2. **Unit Tangent Vector** \( \mathbf{T} \): \( \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \)
3. **Curvature** \( \kappa \):
   a. \( \kappa = \frac{|d\mathbf{T}|}{ds} = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} \)
   b. For a function \( y = f(x) \): \( \kappa = \frac{|f''(x)|}{\left[1 + (f'(x))^2\right]^{3/2}} \)
4. **Principal Unit Normal Vector** \( \mathbf{N} \): \( \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} \)
   a. \( \mathbf{N} \) is perpendicular to the unit tangent vector \( \mathbf{T} \) and points in the direction in which the curve is bending.
   b. **Using acceleration**: \( \mathbf{N} = \frac{\mathbf{a} - \mathbf{a}_T \mathbf{T}}{a_N} \)
5. **Binormal Vector**: \( \mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) \)
6. **Normal and Osculating Planes**:
   a. The **normal plane** is the plane that contains \( \mathbf{N} \) and \( \mathbf{B} \).
   b. The **osculating plane** contains \( \mathbf{T} \) and \( \mathbf{N} \).
   c. The circle in the osculating plane that most approximates the curve at the point \( C \) (same curvature and tangent and its center lies along \( \mathbf{N} \)) is called the **osculating circle**. Its radius \( \rho \) is called the **radius of curvature** and is \( \rho = \frac{1}{\kappa} \) the reciprocal of the curvature of the curve at point \( C \).
7. **Acceleration** \( \mathbf{a} = \frac{dv}{dt} \mathbf{T} + \kappa v^2 \mathbf{N} = a_T \mathbf{T} + a_N \mathbf{N} \)

a. **Tangential Component of Acceleration**: \( a_T = \frac{dv}{dt} = \frac{v \cdot \mathbf{a}}{v} = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{\left| \mathbf{r}'(t) \right|} \)

b. **Normal Component of Acceleration**: \( a_N = \kappa v^2 = \frac{\left| \mathbf{r}'(t) \times \mathbf{r}''(t) \right|}{\left| \mathbf{r}'(t) \right|} \)